A Two-User Variational Mode Decomposition Algorithm for Blind Source Separation of Arbitrary Signals

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***Abstract*—Blind source separation (BSS) is a problem wherein two unknown source signals that have been combined in some fashion when collected at a single receiver must be separated. This problem is exacerbated by source signals that are time-varying, i.e. non-stationary, in nature. All commonly used techniques, such as Fourier analysis, wavelet analysis, and adaptive filtering algorithms do not apply. First, there is no training data, the data may not be digital in nature but have a random, time-varying amplitude, and there are not enough samples to estimate the signal due to the non-stationarity. A novel technique known as empirical mode decomposition (EMD) overcomes many of the issues related to non-stationarity, but suffers in noisy channels. More recently, a technique known as variational mode decomposition (VMD) was introduced that overcomes the limitations of the other techniques to reconstruct unknown, non-stationary signals; this is termed the single user (SU) VMD algorithm. In this paper, we describe a two-user VMD algorithm that improves performance over the SU algorithm by up to an order of magnitude for the problem of BSS. The two-user VMD only assumes that an estimate of the power levels of the two users may be obtained. Performance is most improved when the powers of the two signals are close to equal.**

Keywords—Blind source separation (BSS), empirical model decomposition (EMD), non-stationary, variational mode decomposition (VMD)

I. INTRODUCTION

Signal estimation is a challenging problem, especially in an unknown environment, with non-stationary signals that are random and lack known training data. Two novel techniques have been developed to perform signal reconstruction under such conditions, far surpassing the performance of conventional methods, such as Fourier analysis, wavelet processing, principal components analysis (PCA), singular value decomposition (SVD), or other signal processing methods that require training or stationarity. The first method, called empirical mode decomposition (EMD), constructs a signal using a series of intrinsic mode decompositions, by sequentially extracting frequency components. This is done by an averaging process of the envelopes constructed by the local minima and maxima of the signals [6]. The second method, termed variational mode decomposition (VMD) improves upon the EMD, which suffers if errors occur due to noise or sampling rates [2]. VMD forms a signal from a set of modes, where each mode is selected to be narrowband about some center frequency, which is computed along with the mode [2].

In [2], the authors demonstrated superior performance of VMD compared to EMD in reconstructing a signal in the presence of noise. In [1], VMD was applied to separation of a sinusoidal signal from speech, using PCA also, but the application was limited to this single use case. The issue of selection of the number of modes K in the reconstruction was not addressed in [2]. However, it was addressed more recently in [9], where selection of the number of modes, K, of the VMD algorithm is discussed for several test cases. Also, [9] applies VMD to the problem of blind source separation (BSS). It applies VMD directly to estimate a stronger signal, termed the interferer, first. Then, the interferer estimate is removed from the collected signal to estimate the other signal, termed the signal-of-interest (SOI). It further discusses a modification to the VMD algorithm, wherein certain modes of the decomposition are removed in estimating the interferer.

This process, termed culling, is based on an estimate of the SOI's amplitude or frequency, so that spurious modes that correspond to the SOI can be removed from the interferer modes to produce better estimates of both signals. This is shown to provide an improvement in MSE of the interferer and hence also of the SOI, which is obtained by subtracting the interferer estimate from the received signal [9]. However, the culling approach suffers when the SOI is much weaker in power than the interferer, because all modes estimate the interferer, making it difficult to cull the SOI. In this paper, we describe a two-user version of the VMD algorithm where the SOI is estimated along with the interferer to address the blind BSS problem for non-stationary signals; this two-user algorithm overcomes the limitations of the culling approach, and also performs well when the two signals are at or near equal power.

An outline of the paper is as follows: Section II describes the signal model and problem. Section III discusses conventional techniques that could be applied to BSS, including the original, single user (SU) VMD algorithm introduced in [2] for signal reconstruction and applied in [9] to BSS. We compare also to the culling algorithm in [9]. We also describe more traditional methods of singular value decomposition (SVD) and empirical mode decomposition (EMD). Section IV describes the proposed two-user method. Section V presents simulation results comparing performance of all the algorithms. Finally, conclusions and remarks on further work are given in Section VI.

II. SIGNAL MODEL

We have a received signal, modeled as

$$y(t) = A_x x(t) + I(t) + \sigma_n n(t), \quad (1)$$

where x(t) is the signal-of-interest (SOI), I(t) is an interferer, and n(t) is noise, modeled as additive white Gaussian noise (AWGN). The goal is to jointly estimate both the SOI and the interferer in the presence of the noise, hence this is a BSS problem. Without loss of generality, we assume that the amplitude of I(t) is unity, and we vary the amplitude of the SOI, A_x, alone. We set the SOI amplitude to achieve a particular SOI to interferer power ratio, denoted as a carrier-to-interference (CIR), in dB. A_x is computed from CIR using

$$A_x = \sqrt{10^{CIR/10}}. \quad (2)$$

Without loss of generality, we assume that $A_x < 1$, meaning the SOI is weaker in power than the interferer; we limit our attention, therefore, to the case where $-10 < CIR < 0$ dB, resulting in $0.316 < A_x < 1$. The noise standard deviation, σ_n is computed from the desired carrier-to-noise ratio (C/N), also in dB, written as

$$\sigma_n = \sqrt{10^{-(C/N)/10} \cdot 0.5}\,. \quad (3)$$

We restrict attention to typical scenarios where $0 < C/N < 15$ dB, which results in $0.707 < \sigma_n < 4$. The goal is to extract both the interferer I(t) and the SOI x(t) from the received signal y(t); again, we note that these signals are time-varying, and we do not have training data. First, we present some conventional methods for separating the signals, including the single user (SU) VMD algorithm, SVD, and EMD; we then discuss the proposed two-user VMD.

III. CONVENTIONAL METHODS

A. *Variational Mode Decomposition*

The VMD was originally proposed in [2], as a way to reconstruct a non-stationary signal x(t) as a series of K modes, wherein the k^{th} mode $u_k(\omega_k)$, in the frequency domain, is constructed using a narrowband Wiener filter about a chosen center frequency, ω_k. Modes are selected based on minimization of the bandwidth about ω_k, resulting in a constrained problem written as

$$\underset{u_k,\omega_k}{arg\ min} ||\partial_t[(\delta(t) + \frac{j}{\pi t}) * u_k(t)]e^{-j\omega_k t}||^2$$
$$s.t. \sum_{k=1}^{K} u_k(t) = x(t), \quad (4)$$

where k = 1, 2, …, K. The term in parenthesis is the Hilbert Transform of $u_k(t)$, * denotes convolution, the exponential term translates the component ω_k to baseband, and the partial derivative ∂t is the gradient, used to estimate the bandwidth. Finally, the norm of the entire term is computed using $||\cdot||^2$. This problem is solved by applying two additional constraints. The first constraint is a parameter, α, used to adjust the weight of the first term in Eq. (4) depending on the strength of the noise; larger noise results in a smaller α. The second constraint is a Lagrange multiplier (λ) used to adhere strictly to the constraint dictated by the second term in Eq. (4). The result is the new problem formulation

$$\underset{u_k,\omega_k}{arg\ min} \ \ \alpha||\partial_t[(\delta(t) + \frac{j}{\pi t}) * u_k(t)]e^{-j\omega_k t}||^2$$
$$+||x(t) - \sum_{k=1}^{K} u_k(t) + \frac{\lambda(t)}{2}||^2. \quad (5)$$

After some mathematical manipulation (details provided in [2]), the iterative solution for computing the modes is as follows: initialize n = 0, $u_k^0 = \lambda^0 = \omega_k^0 = 0$, and compute the modes and frequencies as

$$u_k^{n+1}(\omega) = \frac{\hat{X}(\omega) - \sum_{j<k} \hat{u}_j^{n+1}(\omega) - \sum_{j>k} \hat{u}_j^{n}(\omega) + \frac{\lambda^n(\omega)}{2}}{1 + 2\alpha(\omega - \omega_k^n)^2}, \quad (6)$$

$$\omega_k^{n+1} = \frac{\int_0^\infty \omega|\hat{u}_k^{n+1}(\omega)|^2 d\omega}{\int_0^\infty |\hat{u}_k^{n+1}(\omega)|^2 d\omega}, \quad (7)$$

where

$$\lambda^{n+1}(t) = \lambda^n(t) + \tau[x(t) - \sum_{k=1}^{K} u_k^{n+1}(t)], \quad (8)$$

and where τ is a time step constant (may be zero). We iterate from n = 1 to N-1 until the modes and frequencies converge.

We input y(t) to the VMD algorithm, and the output modes are used to compute the estimate of I(t), denoted $\hat{I}(t)$ as

$$\hat{I}(t) = \sum_{k=1}^{K} u_k(t). \quad (9)$$

This equation is based on the assumption that the interferer power is higher, hence it will be estimated first by VMD. We then estimate the SOI as

$$\hat{x}(t) = y(t) - \hat{I}(t). \quad (10)$$

Finally, performance is determined by computing the mean-square error (MSE) between the true signals and their estimates, using

$$MSE_{I(t)} = \overline{((I(t) - \hat{I}(t))^2)}, \quad (11)$$

and

$$MSE_{x(t)} = \overline{((x(t) - \hat{x}(t))^2)}, \quad (12)$$

Note that an issue with the VMD algorithm is in selection of the number of modes, K, similar to selection of the number of signals when using PCA. If under selected, the interferer is not fully captured and errors in estimating the SOI occur. If overestimated, then partial SOI cancellation occurs, again resulting in errors in estimating the SOI. In [9], it is discussed how K can be selected based on the types of signals expected.

When the two signals are near or at equal power, this algorithm will suffer as it does not take into account the presence of the second signal, the SOI; hence, we call this the single user (SU) VMD algorithm. The reconstruction error that results due to the lack of knowledge of the SOI motivates the need to develop an improved two-user version of this algorithm.

B. Variational Mode Decomposition with Mode Culling

An improved SU VMD algorithm using mode culling is presented in [9]. This involves an initial estimate of the frequency bands at which a strong SOI component is present. Then, the modes u_k at those frequencies, ω_k are removed from Eq. (9) to improve the interferer estimate. This is written as

$$\hat{I}(t)_{culled} = \sum_{\substack{k=1 \\ k \neq j, \omega_j \epsilon X(\omega)}}^{K} u_k(t), \tag{13}$$

where we do not include any mode j, where ω_j is estimated to be a frequency of $X(\omega)$, and $X(\omega) = FFT\{x(t)\}$. Note that the ω_j 's have to be estimated or guessed at, since we do not know $X(\omega)$. This can be estimated in practice, e.g. by knowing what bands a particular SOI is in. The SOI is still indirectly obtained from the new interferer estimate using Eq. (10), and MSEs are computed as before from Eqs. (11) and (12).

An alternate way to do the culling is to estimate the amplitude of the SOI, and then remove the $u_k(t)$'s whose amplitude is closest, i.e. the k^{th} component where $A_x \approx \max\{u_k(t)\}$. We can write

$$\hat{I}(t)_{culled} = \sum_{\substack{k=1 \\ k \neq j, u_j(t) \epsilon A_x \approx max(u_j(t))}}^{K} u_k(t). \tag{14}$$

Removal of the estimated SOI modes by culling produces a better estimate of the interferer, which in turn results in an improved estimate of the SOI.

C. Singular Value Decomposition

A technique, which is similar to the well-known method of PCA, is known as singular value decomposition (SVD). It constructs a signal from a subset of its eigenvalues and eigenvectors ([3] and [4]). The idea is that noise is eliminated by choosing a subset that only spans the signal space and does not include the noise space. Suppose we form a received data matrix Y from our received signal y(t), which may be formed from many observations of the received signal. Thus, Y has dimension M x N, where M is the number of observations of the sampled statistical process, and N is the number of samples collected in each observation. We can compute the SVD of the matrix Y as [8]

$$\mathbf{Y} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H, \tag{15}$$

where **U**, **Λ**, and **V** are M x N left singular, N x N diagonal singular, and N x N right singular matrices of **Y**, respectively. Matrix **V** contains the eigenvectors of

$$\mathbf{R}_{YY} = \mathbf{Y}^H\mathbf{Y}, \tag{16}$$

where $\mathbf{R}_{YY}$ has been defined as the covariance matrix of **Y**. The column vectors of U form an orthonormal basis of $\mathbf{R}_{YY}$. To estimate the strong interferer, we form a subset D, D < N, of these vectors, e.g.

$$\mathbf{Y}_D = \sum_{i=1}^{D} \lambda_i \mathbf{U}_i \mathbf{V}_i^H, \tag{17}$$

and then compute the estimate of the interferer as

$$\hat{I}(t) = \mathbf{Y}_{D,i}, \tag{18}$$

where $\lambda = \text{diag}(\Lambda)$, and λ_i is the i^{th} element of the vector λ; similarly, $\mathbf{Y}_{D,i}$, $\mathbf{U}_i$, and $\mathbf{V}_i$ denote the i^{th} column of matrices $\mathbf{Y}_D$, **U** and **V**, respectively. By choosing a set of D basis vectors that correspond to the D largest λ_i's, a reduced rank subspace is formed. Since we typically do not know a priori what value to select for D, we simply choose it to be the number of non-zero eigenvalues. We then estimate the SOI using Eq. (10), and the MSEs using Eqs. (11) and (12), as with the VMD. However, while this algorithm generally performs slightly better than PC because it is not as sensitive to the selection of D being less than the rank of the subspace, it suffers from computational complexity issues [5]. More importantly, the SVD metric is also dependent on eigen-decomposition and estimation of the covariance matrix $\mathbf{R}_{YY}$. Furthermore, it does not lend itself easily to handling a two-user BSS problem.

D. Empirical Mode Decomposition

The EMD decomposes a signal into what is called intrinsic mode functions (IMFs). These are functions that take the form [2]

$$u_k(t) = A_k(t) \cdot cos(\phi_k(t)), \tag{19}$$

where $A_k(t)$ and $\phi_k(t)$ are time-varying amplitude and phase components. The IMFs are computed by a repetitive sifting process, until only a residual component remains. This is described in [6] and [10] and summarized next:

First, we find local maxima and minima; we use these to compute the upper and lower envelopes of the signal, denoted u(t) and l(t), respectively. Initializing index k = 1, we compute the mean as

$$m_k(t) = \frac{u(t)+l(t)}{2}. \tag{20}$$

We compute the sifted IMF,

$$h_k(t) = y(t) - m_k(t). \tag{21}$$

We use this to update the signal

$$y_k(t) = h_k(t) \tag{22}$$

Now we increment k, compute new envelopes of the updated signal, and repeat the calculations in Eqs. (20)-(22) until all IMFs are computed. This completes the first part of the algorithm. In the second part, we set

$$c_1(t) = h_k(t), \tag{23}$$

the last IMF, which represents the highest frequency component present in the signal. We compute a residue

$$r_1(t) = y(t) - c_1(t). \tag{24}$$

We update the signal using the computed residue

$$y_k(t) = r_1(t) \tag{25}$$

Next, we repeat and compute the residues for the remaining IMFs

$$r_j(t) = r_{j-1}(t) - c_j(t), \tag{26}$$

for j = 2, 3, …, n until $r_n(t)$ is a monotonic function. Finally, the reconstructed signal, assumed once again to be the interferer, can be written as

$$\hat{I}(t) = \sum_{j=1}^{n} c_j(t) + r_n(t). \tag{27}$$

Note that $r_n(t)$ represents the mean of y(t). With the interferer estimate having been computed, the SOI and MSEs are computed as with the VMD and SVD algorithms, using Eqs. (10)-(12). As with the preceding algorithms, this algorithm is not suited to handle multiple users. So, we turn our attention back to the VMD and develop a two-user modification of the SU VMD.

IV. Two-User Variational Mode Decomposition (VMD) Algorithm

Referring to Eqs. (1) and (4), the two-user constrained problem may be written as

$$\underset{\substack{u_{k,1},u_{k,2}\\ \omega_k}}{arg\ min}\ ||\partial_t[(\delta(t)+\frac{j}{\pi t})*(u_{k,1}(t)+\hat{A}_x u_{k,2}(t))]e^{-j\omega_k t}||^2,$$
$$s.t. \sum_{k=1}^{K}[u_{k,1}(t)+\hat{A}_x u_{k,2}(t)] = x(t), \tag{28}$$

where $\hat{A}_x$ is an estimate of the amplitude of the SOI, and we introduce a second set of modes for the weaker SOI, $u_{k,2}(t)$. The modes $u_{k,1}(t)$ are for the stronger interferer. Applying the same constraints as for the single user (SU) case, we obtain

$$\underset{\substack{u_{k,1},u_{k,2}\\ \omega_k}}{arg\ min}\ \alpha||\partial_t[(\delta(t)+\frac{j}{\pi t})*(u_{k,1}(t)+\hat{A}_x u_{k,2}(t))]e^{-j\omega_k t}||^2$$
$$+||x(t)-\sum_{k=1}^{K}(u_{k,1}(t)+\hat{A}_x u_{k,2}(t))+\frac{\lambda(t)}{2}||^2. \tag{29}$$

Performing similar mathematical manipulations as in [2] for the SU VMD (details shown in the Appendix), we obtain the modes for both users as

$$u_{k,1}^{n+1}(\omega) = \frac{\hat{X}(\omega)-\sum_{j<k}\hat{u}_{j,1}^{n+1}(\omega)-\sum_{j>k}\hat{u}_{j,1}^{n}(\omega)+\frac{\lambda^n(\omega)}{2}}{1+2\alpha(\omega-\omega_k^n)^2}, \tag{30}$$

$$u_{k,2}^{n+1}(\omega) = \frac{\hat{X}(\omega)-\sum_{j<k}\hat{A}_x\hat{u}_{j,2}^{n+1}(\omega)-\sum_{j>k}\hat{A}_x\hat{u}_{j,2}^{n}(\omega)+\frac{\lambda^n(\omega)}{2}}{\hat{A}_x+2\alpha(\omega-\omega_k^n)^2}, \tag{31}$$

where ω_k is calculated as in Eq. (7), and

$$\lambda^{n+1}(t) = \lambda^n(t)+\tau[x(t)-\sum_{k=1}^{K}(u_{k,1}^{n+1}(t)-\hat{A}_k u_{k,2}^{n+1}(t))], \tag{32}$$

Note the presence of the term $\hat{A}_x$, the estimate of the SOI amplitude, indicating that this term is required to compute the modes for the SOI. Based on the above equations, we obtain the estimates of the interferer and SOI with the two user algorithm as

$$\hat{I}_{Two-User}(t) = \sum_{k=1}^{K} u_{k,1}(t), \tag{33}$$

and

$$\hat{x}_{Two-User}(t) = \sum_{k=1}^{K} u_{k,2}(t). \tag{34}$$

We can therefore compute the MSEs for the two-user algorithm as

$$MSE_{I_{Two-User}(t)} = \overline{((I(t) - \hat{I}_{Two-User}(t))^2)}, \tag{35}$$

and

$$MSE_{x_{Two-User}(t)} = \overline{((x(t) - \hat{x}_{Two-User}(t))^2)}, \tag{36}$$

In the next section, we perform simulations to compare the performance of the SU VMD and two-user VMD algorithms, as well as the conventional SVD and EMD algorithms, comparing the MSEs and plotting the estimated signals.

V. Simulations

We assume the SOI is a chirp signal that takes the form

$$x(t) = e^{j2\pi f_d t} e^{j\pi\beta t^2}, \tag{37}$$

where f_d is the initial frequency of the chirp, and β is the rate of change in the frequency in Hz/sec. We further assume the interferer is a speech signal, modeled as

$$I(t) = \sum_{i=1}^{N_s} s_i(t), \tag{38}$$

where

$$s_i(t) = a_i(t)cos(2\pi[f_{c_i} + \int_0^t f_i(\tau)d\tau] + \theta); \tag{39}$$

and where $a_i(t)$ is the amplitude, which will be time-varying, i.e. non-stationary, f_{ci} is the center frequency of the signal, $f_i(t)$ is the frequency modulation, and N_s is the number of signals combined to form the speech signal [7]. For simulation purposes, we let f_d = 40 kHz, β = 10 kHz/s, and 1 Hz < fc < 1 kHz. The interferer and SOI are shown in Fig. 1. We now study the performance of the two-user VMD algorithm and compare it to the others.

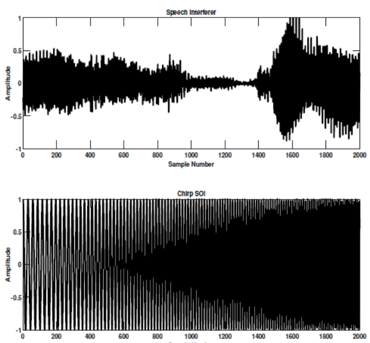

Fig. 1. Speech Interferer and Chirp SOI; f_d = 40 kHz, β = 10 kHz/s, and 1 Hz < fc < 1 kHz

Note that the two-user VMD requires estimation of the amplitude of the SOI, Ax. Hence, it is important to understand how its performance will degrade if there are errors in the estimate.

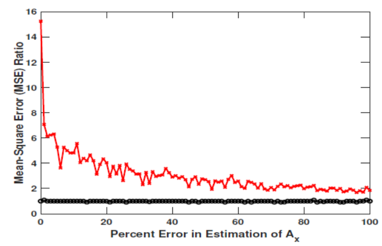

Fig. 2. MSE Ratio of Two-User VMD to Single User (SU) VMD Algorithms vs. Percent Error in Estimation of Ax; Speech Interferer and Chirp SOI; C/N = 10 dB, CIR = 0 dB, K_1 = 6, K_2 = 3

In Fig. 2. we show the MSE ratio of the two-user VMD algorithm to the SU VMD algorithm as a function of percentage error in A_x. In this example, C/N = 10 dB, and CIR = 0 dB, so A_x = 1. From this plot, we see that up to about 20% error in the amplitude can occur before the two-user VMD algorithm degrades to the point where the SU VMD algorithm would be just as effective.

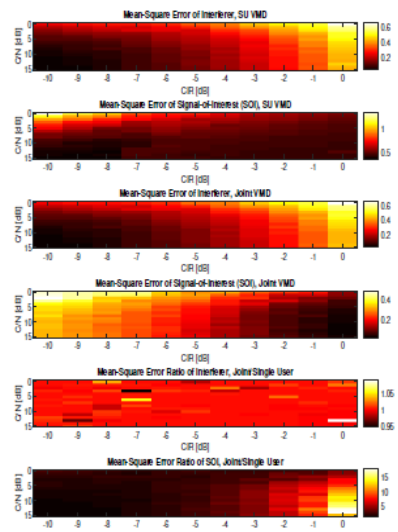

Fig. 3. MSEs of Single User (SU) VMD Algorithm, Two-User VMD Algorithm, and Ratio of Two-User to SU Algorithms; Speech Interferer and Chirp SOI; K_1 = 6, K_2 = 3

Fig. 3. shows the mean-square error (MSE) of the interferer and SOI for the SU VMD algorithm in the top two plots and the two-user VMD in the next two plots. The ratio of MSEs for the two-user and SU algorithms is shown in the last two plots. Notice that the two-user VMD algorithm does not improve the MSE of the interferer, which is the stronger signal. However, it significantly improves the MSEs of the lower powered SOI. Improvement is significant when -5 < CIR < 0 dB, and most significant when CIR approaches 0 dB. This is due to the two-user VMD algorithm taking into account the SOI, and its amplitude, to estimate it, whereas the SU VMD does not. This is the range of CIR where most improvement is needed, because when CIR is near -10 dB, the SOI is weak, so the SU VMD is sufficient.

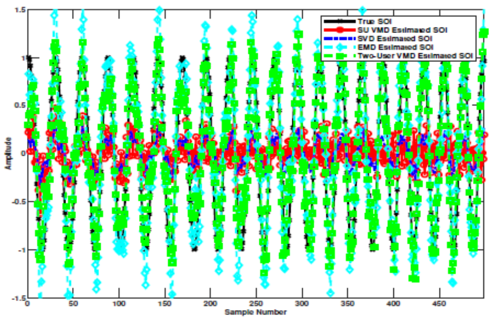

Fig. 4. Estimate of Chirp SOI; Speech Interferer and Chirp SOI; Comparison of SU VMD, SVD, EMD, and Two-User VMD; C/N = 10 dB, CIR = 0 dB, $K_1 = 6$, $K_2 = 3$

Fig. 4. plots the estimate of the SOI for the SU VMD, SVD, EMD, and the proposed two-user VMD algorithms, comparing them to the true chirp SOI, in the presence of the random speech interferer. We show only the first 500 samples to make it more visible. We have set C/N = 10 dB and CIR = 0 dB in this example. Because the SOI power is equal to the interferer power, we would expect the SU algorithm to perform poorly, since it is operating without knowledge of the presence of a second signal. The SVD algorithm also fails, because of a lack of knowledge of the presence of a second signal. The two-user VMD algorithm, however, is able to extract the SOI successfully. The MSEs are calculated from the samples as $MSE_{SU\ VMD} = 0.44$, $MSE_{SVD} = 0.60$, $MSE_{EMD} = 0.08$, and $MSE_{two\text{-}user\ VMD} = 0.03$, reflecting the improved two-user VMD algorithm performance. This is an improvement of 11.1, 12.5, and 3.7 dB over the SU VMD, SVD, and EMD, respectively. The EMD algorithm performs well too, but not as well as the two-user VMD. While EMD is able to estimate the frequency structure of the SOI nearly as well as the two-user VMD, it has errors in estimating the SOI amplitude, due to the interferer. This results in fluctuations that increase its MSE.

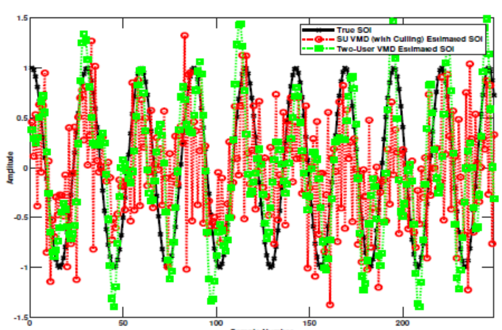

Fig. 5. Estimate of Chirp SOI; Speech Interferer and Chirp SOI; Comparison of Culled SU VMD and Two-User VMD; C/N = 10 dB, CIR = -10 dB, $K_1 = 6$, $K_2 = 3$

Fig. 5 plots the SOI estimate for C/N = 10 dB and CIR = -10 dB to compare the improvement of the two-user algorithm vs. the culling VMD algorithm presented in [9]. Here, the last two modes are culled, per [9]. As mentioned above and discussed in [9], the culling VMD approach suffers at low CIR due to the inability to estimate the weak SOI modes and remove them from the interferer estimate. Here, we see that the two-user VMD approach works better, because it directly uses the SOI's amplitude estimate to compute the SOI from the received signal. MSE estimates are $MSE_{SU\ VMD,\ Culling} = 0.59$ and $MSE_{two\text{-}user\ VMD} = 0.21$, an improvement of 4.5 dB. The SVD and EMD algorithms fail, producing MSEs > 0.8, so these are not shown.

VI. Conclusion

This paper presents a two-user, blind source separation (BSS) algorithm to simultaneously estimate two unknown signals in a time-varying environment. This is done without any knowledge of the signals, i.e. no known sequences or training data, other than an estimate of the relative amplitudes of the two signals. The proposed method is a two-user version of the previously introduced variational mode decomposition (VMD) algorithm, where the modes that reconstruct the signals are computed using their amplitude estimates. Performance results showed an order of magnitude improvement of the two-user VMD algorithm over the single user (SU) case, both with and without mode culling, wherein the stronger signal is estimated with the VMD and the weaker one is obtained by subtracting this estimate from the received signal. We also demonstrated improvement in performance of the two-user VMD over conventional single user (SU) methods including singular value decomposition (SVD) and empirical mode decomposition (EMD). Improvement of the two-user VMD is most pronounced when the powers of the two signals are close, i.e. the CIR is near 0 dB; this is where all the other algorithms suffer because they do not take into account any knowledge of a second signal. We also analyze performance degradation as a function of errors in estimating the signal amplitudes, determining that up to a 20% error is tolerable before performance degrades to where the other algorithms perform just as well. Future work includes development of a truly joint algorithm where both signals may be estimated together, although this is expected to pose significant challenges.

VII. Appendix: Detailed Derivation of Second User Modes

To derive the modes, we start with the constrained problem from Eq. (29), repeated here for convenience as

$$\underset{\substack{u_{k,1},u_{k,2}\\ \omega_k}}{arg\ min}\ \alpha\|\partial_t[(\delta(t)+\frac{j}{\pi t})*(u_{k,1}(t)+\hat{A}_x u_{k,2}(t))]e^{-j\omega_k t}\|^2$$

$$+\|x(t)-\sum_{k=1}^{K}(u_{k,1}(t)+\hat{A}_x u_{k,2}(t))+\frac{\lambda(t)}{2}\|^2. \tag{40}$$

We first exploit Parseval's theorem, which states that the magnitude squared of a function integrated over all time is equivalent to its magnitude squared integrated over all frequency, to translate the above to the frequency domain,

$$\underset{\substack{u_{k,1},u_{k,2}\\ \omega_k}}{arg\ min}\ \alpha||j\omega[(1+sgn(\omega+\omega_k))\cdot (u_{k,1}(\omega+\omega_k)+\hat{A}_x u_{k,2}(\omega+\omega_k))]||^2 + ||\hat{X}(\omega)-\sum_{k=1}^{K}(u_{k,1}(\omega)+\hat{A}_x u_{k,2}(\omega))+\frac{\lambda(\omega)}{2}||^2. \quad (41)$$

Following [2], replace ω with ω - ω_k in the first term of the above equation to get

$$\underset{\substack{u_{k,1},u_{k,2}\\ \omega_k}}{arg\ min}\ \alpha||j(\omega-\omega_k)[(1+sgn(\omega))\cdot(u_{k,1}(\omega)+\hat{A}_x u_{k,2}(\omega))]||^2 + ||\hat{X}(\omega)-\sum_{k=1}^{K}(u_{k,1}(\omega)+\hat{A}_x u_{k,2}(\omega))+\frac{\lambda(\omega)}{2}||^2. \quad (42)$$

Note that we do not have to do this with the second term as all the terms use the variable ω, so a change of terms does nothing. Next, we exploit the fact that real signals in the frequency domain are symmetric about the zero frequency, so we can write each term as twice the integral of the term over positive frequencies only. This further allows us to simplify, since sgn(ω) = 1 for positive ω. We therefore obtain

$$\underset{\substack{u_{k,1},u_{k,2}\\ \omega_k}}{arg\ min}\ \int_0^\infty \{4\alpha(\omega-\omega_k)^2|u_{k,1}(\omega)+\hat{A}_x u_{k,2}(\omega)|^2 + 2|\hat{X}(\omega)-\sum_{k=1}^{K}(u_{k,1}(\omega)+\hat{A}_x u_{k,2}(\omega))+\frac{\lambda(\omega)}{2}|^2\}d\omega. \quad (43)$$

Now, treating the two signals as independent from one another, we solve the minimization problem with respect to the second user first, taking the partial derivative $\partial/d_{uk,2}$ and setting the result to zero. This gives

$$8\alpha(\omega-\omega_k)^2\hat{A}_x u_{k,2}(\omega)-4\hat{A}_x(\hat{X}(\omega)+\hat{A}_x\sum_{k=1}^{K}u_{k,2}+\frac{\lambda(\omega)}{2})=0. \quad (44)$$

Separating the kth term in the summation of the second term, combining it with that of the first term, i.e. $u_{k,2}(\omega)$, and including the iteration index n to account for the most recent updates, we obtain

$$(2\alpha(\omega-\omega_k)^2+\hat{A}_x)u_{k,2}^{n+1}(\omega) -(\hat{X}(\omega)+\sum_{j<k}\hat{A}_x u_{j,2}^{n+1}(\omega)+\sum_{j>k}\hat{A}_x u_{k,2}^{n}(\omega)+\frac{\lambda(\omega)}{2}=0. \quad (45)$$

Solving for $u_{k,2}^{n+1}(\omega)$, we obtain the final result given in Eq. (31), i.e.

$$u_{k,2}^{n+1}(\omega)=\frac{\hat{X}(\omega)-\sum_{j<k}\hat{A}_x\hat{u}_{j,2}^{n+1}(\omega)-\sum_{j>k}\hat{A}_x\hat{u}_{j,2}^{n}(\omega)+\frac{\lambda^n(\omega)}{2}}{\hat{A}_x+2\alpha(\omega-\omega_k^n)^2}, \quad (46)$$

The modes for the stronger interferer may be obtained similarly, as first described in [2]. Here, we can obtain the modes for the first user directly from the modes of the second user, given in Eq. (46). To do this, recall that the amplitude of the first user, the interferer, was set to unity, so we can simply set the variable $\hat{A}_x$ to unity, giving

$$u_{k,1}^{n+1}(\omega)=\frac{\hat{X}(\omega)-\sum_{j<k}\hat{u}_{j,1}^{n+1}(\omega)-\sum_{j>k}\hat{u}_{j,1}^{n}(\omega)+\frac{\lambda^n(\omega)}{2}}{1+2\alpha(\omega-\omega_k^n)^2}, \quad (47)$$

which of course, are the same modes gives in the single user algorithm in [2]. Note that if the interferer had non-unity amplitude, we could instead replace $\hat{A}_x$ with a new variable representing the amplitude estimate of the interferer, $\hat{A}_I$ to obtain its modes.

VIII. Acknowledgment

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